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UNSTEADY AERODYNAMIC CHARACTERISTICS OF A THREE-DIMENSIONAL PLATE CASCADE IN A SUBSONIC GAS FLOW

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The aerodynamic forces on working blades of axial turbomachines in an unsteady subsonic gas flow are often calculated under the hypothesis of cylindrical profiles. According to this hypothesis, the gas flow in every cylindrical layer of a row of blades is replaced by an approximately plane–parallel flow through a periodic airfoil cascade. The range of the cascade and flow parameters for which the hypothesis works is still uncertain. However, the practice of designing high-velocity compressors and fans calls for complete comprehension of the character of the gas flow, in particular, its three-dimensional nature. Since the nonlinear problem on a viscous gas flow through the blade rows is complicated to solve, a manageable linear model of three-dimensional unsteady flow needs to be developed.

The simplest three-dimensional model of an axial turbine is a plate cascade of finite span. The plates are of rectangular planform and are bounded at the ends by parallel planes Π_1 and Π_2 (Fig. 1). This model probably matches most precisely an axial turbine with a tip–root ratio close to unity and a large number of blades. The problem was considered in this statement in [1, 2]. In [1] the method of aerodynamic interference and the mathematical apparatus of Mathieu's functions were used. In [2] the factorization method was used, and good agreement of the numerical results with results of [1] was shown by several examples (for an incompressible liquid). The sophistication of the mathematical model complicates performance of numerical experiment and thereby the qualitative analysis of the results. Moreover, for the methods employed in [1, 2] the geometry of the gas flow region is essential, so they cannot be used to solve the problem within the framework of other models (for example, the model of an annular cascade).

The method of horseshoe vortices (Π-vortices), which is widely used for calculation of incompressible fluid flows, is generalized here for a compressible medium. The method proposed can be used for solving problems of cascade flow in turbines of different geometry, as well as the classical problems of the theory of a finite-span wing.

Statement of the Problem. We shall consider a three-dimensional plate cascade with the stagger δ (Fig. 1), the plates being bounded at the ends by the planes Π_1 and Π_2 (Π_1 is the plane of attachment of blades). The distance between the blades h along the span remains constant.

It is assumed that the cascade blades Σ_n oscillate synchronously according to a certain harmonic law with small amplitude, frequency ω, and constant phase shift μ between the oscillations of adjacent blades. The velocity of the freestream gas flow at infinity ahead of the cascade is V. The distorted gas flow beyond the blade cascade and vortex wakes shedding from each blade owing to the variability of circulation with time and blade height is potential. We shall simulate the vortex wakes W_n by discontinuity surfaces of tangential velocities, which continue the plates behind the trailing edges to infinity.

We introduce Cartesian coordinates $Ox_1y_1z_1$ attached to one of the blades, which we will call the zero-th blade (n = 0). We place the origin at the root chord of the blade lying in the plane of attachment Π_1. We direct the x_1 axis along the chord parallel to the velocity of the main flow, align the y_1 axis with the plate span, and direct the z_1 axis perpendicular to the x_1 axis in the plane Π_1. It is convenient to change to dimensionless coordinates $\bar{x}$, y, and $\bar{z}$ connected with x_1, y_1, and z_1 by the relations

$$x_1 = l\bar{x}, \quad y_1 = ly/\beta, \quad z_1 = l\bar{z}/\beta,$$

where l is the blade height, $\beta^2 = 1 - M^2$; $M = V/a$ is the Mach number; and a is the velocity of sound in the undisturbed flow.

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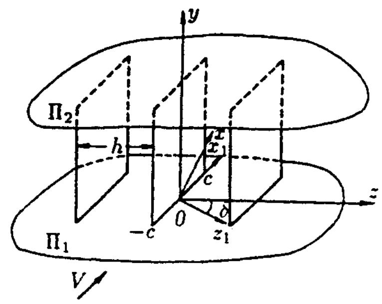

Fig. 1

Under the above assumptions the following problem arises for the amplitude of the potential φ of the perturbed velocities

$$\begin{aligned}
&\Delta\varphi - 2i\frac{M^2}{\beta^2}q\varphi_{\bar{x}} + \frac{M^2}{\beta^2}q^2\varphi = 0,\\
&\varphi_{\bar{z}} = \beta^{-1}v(\bar{x}, y)\exp(in\mu) \;\; (n = 0, \pm 1, \ldots) \quad \text{for } \mathbf{x} \in \Sigma_n,\\
&[p] = [\varphi_{\bar{z}}] = 0 \quad \text{for } \mathbf{x} \in W_n,\\
&\varphi_y = 0 \quad \text{for } \mathbf{x} \in \Pi_1 \cup \Pi_2,\\
&|\nabla\varphi| < \infty \quad \text{for } \bar{x} = 0, 5\lambda^{-1}, \;\; 0 \le y \le \beta.
\end{aligned} \tag{1}$$

Here x = $(\bar{x}, y, \bar{z})$; q = $\omega l/V$; v(x, y) is the prescribed amplitude function of the oscillation velocity (oscillation mode); square brackets denote a sudden increase of the value enclosed in them; i is imaginary unit y, $\lambda = l/(2c)$ is the aspect ratio of the blade, and c is its half-chord.

Equation (1) for φ is obtained by linearization of the problem for uniform flow along the x axis and a dilatation transformation along the y and $\bar{z}$ axes with the parameter $1/\beta$. In addition, the radiation condition is imposed, which implies that solution should not contain waves arriving at the cascade from infinity.

Fundamental Solution. Following [3], to find a fundamental solution we consider the inhomogeneous equation

$$\Delta G - 2iM^2\beta^{-2}qG_{\bar{x}} + M^2\beta^{-2}q^2G = -f(M_0, \bar{x}), \tag{2}$$

where $f(M_0, \bar{x})$ is the given function ($M_0 = (y, \bar{z})$) with the boundary conditions

$$G_y = 0 \quad \text{for} \quad y = 0, \;\; y = \beta.$$

Let us introduce a new unknown function

$$\tilde{G} = e^{iM^2q/\beta^2\bar{x}}G.$$

Then Eq. (2) becomes

$$\Delta\tilde{G} + k^2\tilde{G} = -e^{iM^2q/\beta^2\bar{x}}f(M_0, \bar{x}), \;\; k = Mq/\beta^2.$$

Let us introduce the reference frame (x, y, z) (the z axis is directed along the cascade axis) turned at an angle δ with respect to the initial coordinate system. The coordinates are related by the formulas

$$x = \bar{x}\cos\delta - \bar{z}\sin\delta, \quad z = \bar{z}\cos\delta + \bar{x}\sin\delta.$$

In the new variables we obtain the following equation

$$\Delta\tilde{G} + k^2\tilde{G} = -e^{iM^2q(x\cos\delta + z\sin\delta)/\beta^2}f(M_0, x).$$

With respect to the variable z the fundamental solution should possess generalized periodicity, and according to the boundary condition of nonpenetration it should satisfy the boundary conditions at y = 0 and y = β. Having represented G as a double series, we obtain finally

$$G(\mathbf{x},\mathbf{y}) = \frac{E_\mu}{\bar{h}\beta}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\varepsilon_m}{\Gamma_{mn}}\exp\left[-\Gamma_{mn}|x-\xi|+i\left(\alpha_n+\mu/\bar{h}\right)(z-\zeta)\right]\cos\beta_m y\cos\beta_m\eta. \quad (3)$$

Here $E_\mu = \exp\{iM^2q/\beta^2[(x-\xi)\cos\delta+(z-\zeta)\sin\delta]\}$; $\bar{h} = h/l$; $\Gamma_{mn}^2 = (\alpha_n+\mu/\bar{h})^2+\beta_m^2-k^2$; $\varepsilon_0 = 0.5$; $\varepsilon_m = 1$ with $m = 1, 2, \ldots$; $\alpha_n = 2\pi n/\bar{h}$; $\beta_m = m\pi/\beta$; $y = (\xi, \eta, \zeta)$; Γ_{mn} is chosen such that the radiation condition holds.

However, the calculation of the function G from (3) and its derivatives becomes impossible as $|x-\xi| \to 0$, because when the points coincide, they display singularities, and the series representing them become divergent. Therefore, we should derive another expression for the function G, which should distinguish explicitly these singularities and be free of problems associated with divergence of the series. For this purpose we use the representation [4]

$$\frac{1}{\sqrt{\alpha_n^2+\beta_m^2}}e^{-\sqrt{\alpha_n^2+\beta_m^2}|x-\xi|} = \frac{2}{\pi}\int_0^\infty \cos\beta_m t K_0\left(\alpha_n\sqrt{t^2+|x-\xi|^2}\right)dt$$

(K_0 is a modified Bessel function). Substituting it in (3) and taking into account that

$$\sum_{m=1}^{\infty}\cos mx = -\frac{1}{2}+\pi\sum_{n=-\infty}^{\infty}\delta(x-2n\pi),$$

we write

$$G = \frac{E_\mu}{2\pi N\bar{h}\beta}\sum_{\nu=0}^{N-1}e^{i\nu\mu}\Big\{\sum_{n=-\infty}^{\infty}e^{i\alpha_n(z-\zeta-\nu\bar{h})}\Big\{K_0(a_n r_+(0))+K_0(a_n r_-(0))+$$
$$+\sum_{m=1}^{\infty}\left[K_0(a_n r_+(m))+K_0(a_n r_+(-m))+K_0(a_n r_-(m))+K_0(a_n r_-(-m))\right]\Big\}\Big\}$$

($a_n = \sqrt{\alpha_n^2 - k^2}$ $r_\pm^2(m) = (y \pm \eta + 2m\beta)^2 + (x - \xi)^2$, $\delta(x)$ is the Dirac delta function, N is the number of blades in the main cascade period). It was shown in [5] that with $|\varphi| < 2\pi$

$$\sum_{n=1}^{\infty}\cos n\varphi K_0\left(\sqrt{n^2-k^2}z\right) = \frac{1}{2}\pi\left\{\frac{1}{2}Y_0(kz)+\sum_{n=-\infty}^{\infty}\frac{\cos k\sqrt{z^2+(2n\pi-\varphi)^2}}{\sqrt{z^2+(2n\pi-\varphi)^2}}\right\}.$$

Using this result and taking into account that

$$\sum_{\nu=0}^{N-1}e^{i\nu\mu} = \begin{cases} 0 & \text{for } \mu \neq 0, \\ N & \text{for } \mu = 0 \end{cases}$$

and $K_0(iz) = -1/2\pi i H_0^{(2)}(z)$ with $-\pi/2 < \arg z \leq \pi$, we obtain

$$G = \frac{E_\mu}{\pi}\sum_{\nu=0}^{N-1}e^{i\nu\mu}\left\{\frac{1}{4}\sum_{n=-\infty}^{\infty}\left(\frac{\cos kR_+}{R_+}+\frac{\cos kR_-}{R_-}\right)+\right.$$
$$+\frac{1}{N\bar{h}}\sum_{m,n=1}^{\infty}\cos\alpha_n(z-\zeta-\nu\bar{h})\left[K_0(a_n r_+(m))+\right.$$
$$\left.\left.+K_0(a_n r_+(-m))+K_0(a_n r_-(m))+K_0(a_n r_-(-m))\right]\right\}+A_\mu.$$

Here $A_\mu = 0$ with $\mu \neq 0$;

$$A_0 = \frac{E_\mu}{4h}\left\{Y_0(kr_+(0)) + Y_0(kr_-(0)) - i\sum_{m=-\infty}^{\infty}\left[H_0^{(2)}(kr_+(m)) + H_0^{(2)}(kr_-(m))\right]\right\};$$

Y_0 is a Bessel function of the second kind, $H_0^{(2)}$ is the Hankel function, and $R_\pm^2 = (x - \xi)^2 + (y \pm \eta)^2 + (z - \zeta - \nu\bar{h} - n N \bar{h})^2$.

The series in Hankel functions in A_0 is analogous to the expression obtained by the reflection method at the strip $\{0 \leq y \leq \beta, -\infty < x < \infty\}$ at whose boundaries a zero Neumann condition holds. The function A_0 has a logarithmic singularity when the points x and y coincide. Using Poisson's summation formula

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{2\pi}\sum_{\nu=-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)e^{-i\nu\tau}d\tau$$

and the integral representation of the Hankel functions [7], we obtain

$$G_0 = -\frac{iE_\mu}{4h}\sum_{m=-\infty}^{\infty}\left[H_0^{(2)}(kr_+(m)) + H_0^{(2)}(kr_-(m))\right] =$$

$$= -\frac{E_\mu}{2\bar{h}\beta}\left[\frac{e^{-ik|x-\xi|}}{ik} + 2\sum_{\nu=1}^{\infty}\frac{e^{-\sqrt{(\nu\pi/\beta)^2-k^2}\,|x-\xi|}}{\sqrt{(\nu\pi/\beta)^2-k^2}}\cos\frac{\nu\pi}{\beta}y\cos\frac{\nu\pi}{\beta}\eta\right].$$

Since there is a term tending to infinity as $k \to 0$ in this formula, there is no limiting process in the solution of the Laplace equation. Inasmuch as the fundamental solution is determined to within the solution of the homogeneous equation, one can subtract $(1/ik)\exp ik(x - \xi)$ in the square brackets, thus providing the passage to limit [6]. The convergence of the series with respect to ν can be significantly improved by subtracting and adding the series with $k = 0$, which equals

$$\frac{1}{2}\frac{\pi}{\beta}|x-\xi| - \frac{1}{4}\ln\left\{4\left[\operatorname{ch}\frac{\pi}{\beta}(x-\xi) - \cos\frac{\pi}{\beta}(\eta - y)\right]\left[\operatorname{ch}\frac{\pi}{\beta}(x-\xi) - \cos\frac{\pi}{\beta}(\eta + y)\right]\right\}.$$

In addition, the singularity of the series for G_0 as $|x - \xi| \to 0$ is removed.

Integral Representation for the Velocity of a Gas Flow. Applying the technique reported in [8], one can obtain a representation for the velocity of a gas flow in terms of the values of the surface curl $\gamma = n \times [v]$ (the vortex layer intensity) at the blade surfaces Σ_n and the vortex wakes behind them W_n. Equation (1) can be written as

$$\operatorname{div}\mathbf{U} + \beta^2k^2\varphi = 0,$$

if we introduce the vector **U** with components

$$U_x = \frac{\partial\varphi}{\partial x} - 2i\frac{M^2}{\beta^2}q\varphi, \quad U_y = \frac{\partial\varphi}{\partial y}, \quad U_z = \frac{\partial\varphi}{\partial z}.$$

We introduce the function ψ with continuous second derivatives, which satisfies the equation conjugate to (1). We construct the vector **V** using the formulas

$$V_x = \frac{\partial\psi}{\partial x} + 2i\frac{M^2}{\beta^2}q\psi, \quad V_y = \frac{\partial\psi}{\partial y}, \quad V_z = \frac{\partial\psi}{\partial z}.$$

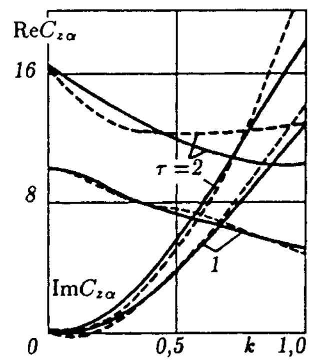

Fig. 2

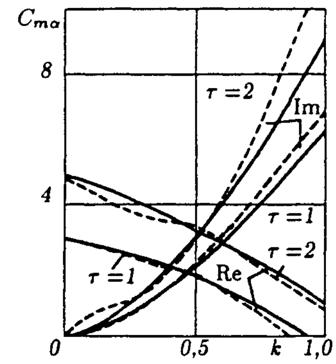

Fig. 3

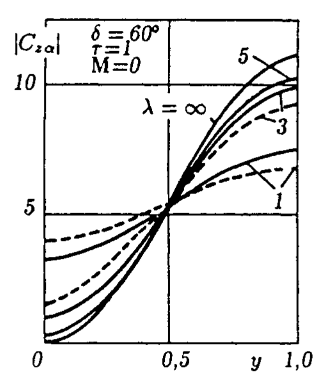

Fig. 4

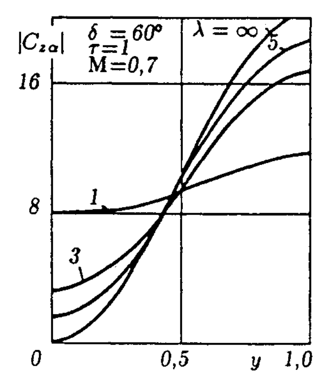

Fig. 5

Then the function ψ satisfies the equation

$$\operatorname{div}\mathbf{V} + \beta^2 k^2 \psi = 0.$$

We integrate the difference

$$\psi(\operatorname{div}\mathbf{U}+\beta^2k^2\varphi)-\varphi(\operatorname{div}\mathbf{V}+\beta^2k^2\psi) = \operatorname{div}(\psi\mathbf{V})-\operatorname{div}(\varphi\mathbf{U})+\nabla\varphi\cdot\mathbf{V}-\nabla\psi\cdot\mathbf{U}$$

over a certain volume Ω wherein the above equations hold. Using the Gauss–Ostrogradskii formula, we obtain

$$\int\limits_{\Omega}(\psi\operatorname{div}\mathbf{U} - \varphi\operatorname{div}\mathbf{V})\,d\Omega = \int\limits_{S}(\psi\mathbf{U} - \varphi\mathbf{V})\mathbf{n}\,dS + 2i\frac{\mathrm{M}^2}{\beta^2}q\int\limits_{S}\varphi\psi\cos(\mathbf{n},x)\,dS,$$

where S is the surface bounding the volume Ω; $\mathbf{n}$ is the outward (with respect to Ω) normal to this surface.

Now let $\psi = G(\mathbf{x}, \mathbf{y})$ be the fundamental solution to Eq. (1). Then for the points $\mathbf{x} \in \Omega$ we obtain the representation of the velocity potential in terms of its value at the boundary S:

$$\varphi = \varphi_0 - \frac{2i\mathrm{M}^2 q\varphi_1}{\beta^2}. \tag{4}$$

Here

$$\varphi_0 = \int\limits_{S}[G(\mathbf{n}\cdot\nabla_y)\varphi - \varphi(\mathbf{n}\cdot\nabla_y)G]dS; \quad \varphi_1 = \int\limits_{S}\varphi G n_\xi dS.$$

We calculate the velocity vector $\mathbf{v} = \nabla\varphi$ using Green's formula (4). Applying the transformations carried out in [8] and the equality

$$\nabla_x G = -\nabla G_y + \mathbf{g},$$

we have

$$\nabla\varphi_0 = -\int_S [(\mathbf{n}\times\mathbf{v})\times\nabla_y G + (\mathbf{n}\cdot\mathbf{v})\nabla_y G - \varphi\mathbf{n}\Delta_y G]\, dS + \int_S [\mathbf{g}(\mathbf{n}\cdot\mathbf{v}) - \varphi(\mathbf{n}\cdot\nabla_y)\mathbf{g}]\, dS,$$

$$\begin{aligned}\nabla\varphi_1 &= -\int_S \varphi\nabla_y G n_\xi\, dS + \int_S \varphi\mathbf{g} n_\xi\, dS = \\ &= -\int_\Omega \nabla_y \left[\frac{\partial}{\partial\xi}(G\varphi)\right] d\Omega + \int_\Omega \frac{\partial}{\partial\xi}(G\nabla\varphi) d\Omega + \int_S \varphi\mathbf{g} n_\xi dS = \\ &= -\int_S \left[\mathbf{n}\frac{\partial}{\partial\xi}(G\varphi) - n_\xi(G\nabla\varphi + \mathbf{g}\varphi)\right] dS.\end{aligned}$$

Substituting the resulting expressions in formula (4), we find finally

$$\begin{aligned}\nabla\varphi = &-\int_S [(\mathbf{n}\times\mathbf{v})\times\nabla_y G + \nabla_y G(\mathbf{n}\cdot\mathbf{v})] dS + \int_S [\mathbf{g}(\mathbf{n}\cdot\mathbf{v}) - \varphi(\mathbf{n}\cdot\nabla_y)\mathbf{g}] dS + \\ &+ 2i\mathrm{M}^2 q/\beta^2 \int_S \left[\left(\mathbf{n}\frac{\partial\varphi}{\partial\xi} - \nabla\varphi n_\xi\right) G - \mathbf{g} n_\xi \varphi\right] dS - k^2\beta^2 \int_S \mathbf{n}\varphi G dS.\end{aligned} \tag{5}$$

The integral representation (5) has been obtained for an arbitrary point x lying inside the region Ω with a sufficiently smooth boundary S. For the problem of flow past a thin body it retains the same form if the integration is carried out over one side of the surface of the body, the vector $\mathbf{v}$ is replaced by its jump $[\mathbf{v}]$ when passing through this surface, and allowance is made for $[\mathbf{v}\cdot\mathbf{n}] = 0$. Here it is convenient to introduce the vector intensity of the vortex layer $\gamma = \mathbf{n} \times [\mathbf{v}]$ (the surface curl). Then $[\mathbf{v}] = \gamma \times \mathbf{n}$, $[\varphi] = \int_{L(M_0)}(\gamma \times \mathbf{n})\cdot d\mathbf{x}$. The line $L(M_0)$ is drawn at the vortex surface Σ from the leading edge of the blade to the point M_0 where the value of the potential is determined.

For the problem under consideration $\mathbf{n} = (0, 0, 1)$, $(\mathbf{n}\cdot\nabla_y)\mathbf{g} = \partial\mathbf{g}/\partial z$, $g_x = g_z = 0$, $\gamma = (\gamma_x, \gamma_y, 0)$, and then

$$\mathbf{v}(\mathbf{x}) = -\int_S \left(\gamma\times\nabla_y G + [\varphi]\frac{\partial\mathbf{g}}{\partial\zeta}\right) dS + \int_S \left(\frac{2i\mathrm{M}^2 q}{\beta^2}[v_\xi] - \beta^2 k^2[\varphi]\right) \mathbf{n} G\, dS. \tag{6}$$

With $M = 0$ and $G = 1/(4\pi|\mathbf{x} - \mathbf{y}|)$ we obtain from (6) the Biot–Savart formula widely used in the theory of wings and cascades.

Projecting the expression for the velocity (6) onto the plane normal to S at the point $\mathbf{x}$ and demanding the fulfillment of the blade nonpenetration condition, we obtain a functional equation for the vector $\gamma(\mathbf{y})$ at points of the surface S. The vector γ involves the following two components at the blade and vortex sheet: $\gamma_y = \gamma_{y+} + \gamma_{y-}$ and γ_x. Here γ_{y+} is the intensity of the bound vortex whose axis is parallel to the y axis; γ_x and γ_{y-} are the intensities of free vortices with the axes along the axes x and y.

The intensities of free vortices can be expressed in terms of the intensity of bound vortices:

for free vortices at the blade

$$\gamma_{y-}(x, y) = \frac{-i\omega}{V}\int_{-c}^{x} \gamma_{y+}(\xi, y) e^{i\frac{\omega}{V}(\xi - x)} d\xi; \tag{7}$$

for free vortices with the axis along the y axis the coordinate of the trailing edge of the blade at the points of the vortex sheet $x = c$ should be the upper limit of the integral in (7);

for free vortices with the axis along the x axis

$$\gamma_x(x,y) = -\int_{-c}^{x} \frac{\partial \gamma_{y+}}{\partial y}(\xi, y) e^{i\frac{\omega}{V}(\xi - x)} d\xi.$$

At the blade the potential jump is determined from the formula

$$[\varphi](x,y) = -\int_{-c}^{x} \gamma_{y+}(\xi, y) e^{i\frac{\omega}{V}(\xi - x)} d\xi$$

while at the wake the upper limit must be substituted for c.

Method of Horseshoe Vortices. The integral presentation (6) admits the construction of a numerical algorithm analogous to that obtained with the method of П-vortices in the theory of a finite-span wing in an incompressible fluid. To this end, we assume that in the region occupied by the gas there is one vortex filament L lying in the plane $z = 0$. We consider its intensity Γ to be constant along L. In this case

$$\gamma(\mathbf{x}) = \Gamma \frac{\partial \mathbf{x}}{\partial s}$$

(s is the arc coordinate of the line L). In our case s is either y or x.

We substitute the cascade blades and vortex wakes behind them for the vortex surfaces and turn from the continuous distribution of vortices to the discrete one. For this purpose we separate the surface of the blade into N_1 strips along y, then each strip into N_2 parts along x. We simulate each resulting tetragon by a П-vortex composed of a part of the bound vortex, which is directed along the y axis and is of size 2δy and intensity $\Gamma_{+i} = lV\Gamma_i$, and the system of free vortices defined in the preceding section.

Using the integral representation (6), we obtain the velocity component normal to the blade, induced by the i-th bound vortex:

$$v_n^i = -\Gamma_i \int_{y_i - \delta y}^{y_i + \delta y} \left(\frac{\partial G}{\partial \xi} \cos\delta + \frac{\partial G}{\partial \zeta} \sin\delta + 2i \frac{M^2 q}{\beta^2} G - \beta^2 k^2 \Delta\xi G \right) d\xi,$$

where $\Delta\xi$ is the length of the side of the element along the x axis.

Since the integrands contain singularities, if the points **x** and **y** coincide, then in calculating the integrals they should be separated so that the integrals of the separated parts can be found analytically. Then

$$\int \frac{\partial G}{\partial \xi} d\eta = \int \left(\frac{\partial G}{\partial \xi} - \frac{x - \xi_i}{4\pi R^3} \right) d\eta + (x - \xi_i) I_1,$$

$$\int \frac{\partial G}{\partial \zeta} d\eta = \int \left(\frac{\partial G}{\partial \zeta} - \frac{z - \zeta_i}{4\pi R^3} \right) d\eta + (z - \zeta_i) I_1,$$

$$\int G d\eta = \int \left(G - \frac{1}{4\pi R} \right) d\eta + I_2.$$

Here $R^2 = (x - \xi_i)^2 + (y - \eta)^2 + (z - \zeta_i)^2$;

$$I_1 = \frac{\eta - y}{4\pi[(x - \xi_i)^2 + (z - \zeta_i)^2]R}; \qquad I_2 = \frac{1}{4\pi} \ln|\eta - y + R|;$$

ξ_i and ζ_i are the coordinates of the i-th bound vortex. The integrals of the functions without singularities are calculated approximately from the formula

$$\int_{y_i-\delta y}^{y_i+\delta y} G_1 d\eta \approx 2\delta y G_1(y_i).$$

The formulas for determining the normal component of velocity induced by free vortices shedding from the i-th bound vortex and vortex layers shedding from the ends of the vortex are derived in the same way. For brevity we write these formulas for the case $\delta = 0$:

for the free vortices

$$v_{n1}^i(\mathrm{x}_j, x_i, \eta) = iq\Gamma_i e^{-iqx_{0ij}}\Big\{2\delta y \int_{-\infty}^{x_{0ij}} e^{iq\xi}\Big[\frac{\partial G}{\partial \xi} - \frac{\xi}{4\pi R_1^3(Y)} +$$

$$+2i\frac{\mathrm{M}^2 q}{\beta^2}\left(G - \frac{1}{4\pi R_1(Y)}\right)\Big]\, d\xi + \frac{1}{4\pi}\int_{-\infty}^{x_{0ij}} \frac{e^{iq\xi} - 1}{\xi}\left[\frac{Y_+}{R_1(Y_+)} - \frac{Y_-}{R_1(Y_-)}\right] d\xi +$$

$$+i\frac{\mathrm{M}^2 q}{2\pi\beta^2}\int_{-\infty}^{x_{0ij}} e^{iq\xi}\ln\left|\frac{Y_+ + R_1(Y_+)}{Y_- + R_1(Y_-)}\right| d\xi + \frac{1}{4\pi}\left[Y_+ I_3(Y_+) - Y_- I_3(Y_-)\right]\Big\},$$

for the free vortex layer

$$v_{n-}^i(\mathrm{x}_j, x_i, \eta) = \Gamma_i e^{-iqx_{0ij}}\left\{\int_{-\infty}^{x_{0ij}}\left[e^{iq\xi}\frac{\partial G}{\partial \eta} + \frac{Y}{4\pi R_1^3(Y)}\right] d\xi + I_4\right\},$$

for the potential jump

$$v_{n2}^i(\mathrm{x}_j, x_i, \eta) = \Gamma_i e^{-iqx_{0ij}}\beta^2 k^2\left\{2\delta y\int_{-\infty}^{x_{0ij}} e^{iq\xi}\left(G - \frac{1}{4\pi R_1(Y)}\right) d\xi +\right.$$

$$\left.+\frac{1}{4\pi}\int_{-\infty}^{x_{0ij}} e^{iq\xi}\ln\left|\frac{Y_+ + R_1(Y_+)}{Y_- + R_1(Y_-)}\right| d\xi\right\}.$$

Here $Y_\pm = Y \pm \delta_y$; $Y = \eta - y_j$; $x_{0ij} = x_j - x_i$; $R_1^2(Y) = \xi^2 + Y^2$;

$$I_3(Y) = \int \frac{d\xi}{\xi R_1(Y)} = -\frac{1}{|Y|}\ln\left|\frac{|Y| + R_1(Y)}{\xi}\right|; \quad \Delta\xi = \frac{1}{\lambda N_2};$$

$$I_4 = \int \frac{d\xi}{R_1^3(Y)} = \frac{\xi}{Y^2 R_1(Y)}.$$

The coordinates of the vortices and control points at the plate are found in a standard way through the method of discrete vortices.

From the fulfillment of the boundary condition of nonpenetration of the cascade plates we obtain the system of algebraic equations for the intensities of bound vortices Γ_i ($i = 1, \ldots, M_1$, $M_1 = N_1 \times N_2$). The right-hand side of the system is governed by the law of oscillations of the cascade blades.

Aerodynamic Characteristics. Having found Γ_i, we determine the pressure differential at the blade with the help of Zhukovskii's theorem "in the small"

$$p_+ - p_- = \rho_\infty V \gamma_{y+},$$

where p_+ and p_- are the limiting values of p when approaching the blade surface from the left and from the right, and ρ_∞ is the freestream gas density.

The coefficients of aerodynamic force C_z and moment C_m with respect to the coordinate origin are determined from the formulas

$$C_z = \frac{2Z}{\rho_\infty V^2 S} = \frac{2}{S}\int_S \gamma_{y+}\,dx\,dy,$$

$$C_m = \frac{M}{\rho_\infty V^2 cS} = \frac{1}{cS}\int_S (\mathbf{x} \times \mathbf{i}_z)\gamma_{y+}\,dx\,dy$$

(S is the blade surface area, $\mathbf{i}_z$ is the unit vector along the z axis, and **M** is the moment).

In discrete form with using the previously determined Γ_i these formulas are as follows:

$$C_z = \frac{4\delta y\lambda}{\beta}\sum_{i=1}^{M_1}\Gamma_i, \qquad C_m = \frac{4\delta y\lambda^2}{\beta}\sum_{i=1}^{M_1}(\mathbf{x} \times \mathbf{i}_z)\Gamma_i.$$

The force and moment acting on any profile along the blade height are found from analogous formulas, but summation is performed within one strip along y.

Discussion of Calculation Results. This algorithm has been used for computer calculations of the unsteady aerodynamic characteristics of the cascades for different values of the governing parameters.

Figures 2 and 3 show the dependence of the coefficients of force $C_{z\alpha}$ and moment $C_{m\alpha}$ relative to the y axis drawn along the leading edge of the blade under torsional oscillations with respect to the leading edges of the blades on the Strouhal number $k = 2\omega c/a$ for a cascade density $\tau = 2c/h$ equal to 1 and 2. In this case the stagger angle $\delta = 0$, the phase shift between the oscillations of the adjacent blades $\mu = \pi$, and the oscillation mode remained invariant with the blade height. The dashed lines present the calculation results obtained in [9] through the two-dimensional theory of subsonic flow past cascade plates. As is seen from the calculations presented, the discrepancy of the results is slight, except for the high densities ($\tau = 2$) and the Strouhal numbers ($k \sim 1$). The comparison shows satisfactory agreement of the calculation results obtained through the method proposed and on the basis of the simpler two-dimensional theory.

Figures 4 and 5 demonstrate the change of the modulus of the coefficient of aerodynamic force per unit span $|C_{z\alpha}|$ along the blade span ($y \to y/l$) with aspect ratios $\lambda = 1, 3, 5$ and ∞ for compressible and incompressible liquids with the phase shift $\mu = \pi$. The calculations were carried out for the cascade with $\tau = 1$, $\delta = 60°$. In this case the Strouhal number $k_1 = \omega c/V$ was assumed equal to 0.5, and the Mach number $M = 0$ and 0.7. The mode of the blade oscillations in the direction of the z axis $v(x, y) = (1 + ik_1x)[1 - \cos(\pi y/l)]$ corresponds to torsional oscillations of the plates relative to their middle with variable torsion amplitude along the y axis.

The case $\lambda = \infty$ corresponds to calculations under the hypothesis of plane transverse profiles, when aerodynamic characteristics for each value of y depend on the law of plate oscillations only in the given profile. The dashed curves in Fig. 4 are plotted on the basis of the results of [1]. The agreement of the results is good. This justifies the conclusion drawn in [1] that the aerodynamic load distribution along the plate span levels off as the plate aspect ratio decreases. According to the data presented, this effect is manifested almost identically for compressible and incompressible liquids, although it was stated in [1] that with $M = 0.7$ the leveling out is more abrupt. In the middle section, curves with different λ intersect at one point both with $M = 0$ and $M = 0.7$, i.e., the calculation results coincide with the data obtained from the two-dimensional theory.

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REFERENCES

1. D. N. Gorelov, V. B. Kurzin, and V. E. Saren, Aerodynamics of Cascades in Unsteady Flow [in Russian], Nauka, Novosibirsk (1971).

2. G. S. Lipovoi, Factorization Method in Hydroaeromechanics Problems [in Russian], Naukova Dumka, Kiev (1977).
3. A. N. Tikhonov and A. A. Samarskii, Equations of Mathematical Physics [in Russian], Nauka, Moscow (1972).
4. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series, Special Functions [in Russian], Nauka, Moscow (1983).
5. L. Lewin, Theory of Waveguides, London, Newness-Butterworths (1951).
6. B. Noble, Methods Based on the Weiner-Hopf Technique for the Solution of Partial Differential Equations, Pergamon Press, London–New York–Paris, (1958).
7. T. N. Galishnikova and A. S. Iliinski, Numerical Methods in Diffraction Problems [in Russian], Moscow University Press, Moscow (1987).
8. V. P. Ryabchenko, "Quasi-two-dimensional approximation in the problem of stationary subsonic flow past a spatial annular cascade," Prikl. Mekh. Tekh. Fiz., No. 2 (1982).
9. D. N. Gorelov, V. B. Kurzin, and V. E. Saren, Atlas of Unsteady Aerodynamic Characteristics of Cascade Profiles [in Russian], Nauka, Novosibirsk (1974).